

# Random Walk on Directed Dynamic Graphs

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**Abstract.** Dynamic graphs have emerged as an appropriate model to capture the changing nature of many modern networks, such as peer-to-peer overlays and mobile ad hoc networks. Most of the recent research on dynamic networks has only addressed the undirected dynamic graph model. However, realistic networks such as the ones identified above are directed. In this paper we present early work in addressing the properties of directed dynamic graphs. In particular, we explore the problem of random walk in such graphs. We assume the existence of an oblivious adversary that makes arbitrary changes in every communication round. We explore the problem of covering the dynamic graph, that even in the static case can be exponential, and we establish an upper bound  $O(d_{max}n^3 \log^2 n)$  of the cover time for balanced dynamic graphs.

## 1 Introduction

Dynamic graphs have emerged as an appropriate model to capture the changing nature of many modern networks, such as peer-to-peer overlays and mobile ad hoc networks. Therefore, the study of dynamic graph models is a topic of practical relevance. Unfortunately, most of the recent research on dynamic networks has only addressed the undirected dynamic graph model [AKL08, KLO10]. This limits the applicability of the results, as many realistic networks are directed. For instance, due to variations in radio hardware, transmission power, interferences, a substantial percentage of wireless links are asymmetric [SAZ07]. Moreover, many overlay peer-to-peer protocols build asymmetric membership views, e.g. [VGvS05].

In this paper we consider the *cover time*, the expected time taken by a random walk to visit every node of the graph at least once. We distinguish two types of random walks: i) A *simple random walk* is a stochastic process that starts at one node of a graph and at each step moves to an adjacent node chosen uniformly at random among all neighbors of a current node; ii) A *lazy random walk* starts at one node of a graph and at each step, either stays in the same node with probability  $k$ , which may depend on the current node, or moves to an adjacent node with probability  $\frac{1-k}{d}$ , where  $d$  is the node out-degree.

The cover time of random walks has been extensively studied for static graphs and bounds for many different classes of static graphs are known in the literature. For a comprehensive survey we refer the interested reader to [Lov93]. However, recent results have shown that bounds for static graph models differ significantly from those for dynamic graphs. It is well known that in undirected graphs the worst-case cover time of a simple random walk is  $O(n^3)$  [AKL<sup>+</sup>79]. On the other hand, [AKL08] has shown that a simple random walk on undirected dynamic graphs can have exponential cover times. The authors of [AKL08] also showed that the cover time of maximum-degree random walk on (non-bipartite) dynamic graph is  $O(d_{max}^2 n^3 \log^2 n)$  where  $d_{max}$  is a maximum node degree of the graph.

The directed dynamic graphs are even more challenging. The cover time for random walks on static directed graphs can be exponential. Aggravated by the presence of the oblivious adversary, we first address the question of the explorability of a dynamic graph. Furthermore, we are interested in determining whether there exist subclasses of directed graphs with polynomial cover time (other than the directed analogues to undirected graphs).

To the best of our knowledge, this is the first work on random walks on directed dynamic graphs with oblivious adversary. Our main contributions are as follows. First we prove that the adversary can prevent the explorability of any periodic dynamic graphs, with period greater than 2. Next we show that by applying a lazy strategy we can circumvent this impossibility result and guarantee that all vertices are reachable by the random walk on the dynamic graph. We then establish an upper bound  $O(d_{max}n^3 \log^2 n)$  for cover time of a lazy random walk with  $k = 1 - \frac{d}{d_{max}+1}$  on balanced dynamic graphs. A graph is called balanced, or Eulerian, when its in-degree is equal to its out-degree.

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The resulting upper bound also tightens the previous bound for undirected dynamic graphs that are a subclass of balanced graphs.

The rest of the paper is organized as follows. In Section 2 we describe the model we use for our results. Section 3 discusses the problem of random walks on dynamic graphs and presents the main results. Finally, Section 4 presents the conclusions and outlines the directions for future work.

## 2 Model

We model a network as a fixed set of nodes that operate in synchronous rounds. The directed edges represent unidirectional communication links from one node to another. Let  $V = \{v_1, v_2, \dots, v_n\}$  be a set of vertices.  $\mathcal{G} = G_1, G_2, \dots$  denote a sequence of directed graphs where  $G_t$  is a static directed graph on  $V$  in round  $t$ . Further, we consider oblivious adversary that can make arbitrary changes in every round as long as the graph remains strongly connected in every round (i.e. there exists a path between each pair of nodes in the graph). We say  $\mathcal{G}$  is strongly connected if each  $G_t$  is strongly connected. This model captures the dynamic nature of connected wireless networks. It can be also used to model static networks where packet losses may occur. A similar model for undirected graphs was introduced in [KLO10].

Note that without further constraints on the topology of the graph, it is easy for the adversary to prevent the random walk from exploring the graph. This can be illustrated with the following example. Consider  $\mathcal{G}$  a strongly connected dynamic graph. A simple random walk starts at the node  $v_i$  that initially has only one neighbor  $v_j$ . In the first round, the random walk moves from  $v_i$  to  $v_j$ . In the next round the adversary changes the network so that now  $v_j$ 's only neighbor is  $v_i$ . So the random walk moves from  $v_j$  back to  $v_i$ . This process can be repeated indefinitely. Even if each graph  $G_t$  in  $\mathcal{G}$  is strongly connected, the random walk can only reach  $v_i$  and  $v_j$ .

## 3 Results

### 3.1 Explorability with Simple Random Walk

We first address the problem of whether a given dynamic graph is explorable with a simple random walk (i.e., there exists a positive probability of a random walk starting in a vertex  $v_i$  to reach all other vertices in the graph). As illustrated by the example above, it is easy to show that without further constraints on the power of the adversary, the dynamic graph can be unexplorable.

We begin by giving a negative result, showing that if the adversary is allowed, in each step, to generate graphs with period  $c > 2$ , then the dynamic graph is unexplorable (in a strongly connected graph, a period of a graph is the greatest common divisor of the lengths of all directed circuits).

To prove this result, we use the following proposition that has been demonstrated in [JS96].

**Proposition 1.** *Let  $G$  be a strongly connected directed graph. If  $G$  has period  $c$  then it can be partitioned into  $c$  sets  $C_0, C_1, \dots, C_{c-1}$  such that*

- (a) *for any  $k < c$ ,  $v_i \in C_k$ , there exists an edge from  $v_i$  to  $v_j$ , such that  $v_j \in C_{(k+1) \bmod c}$ ; and*
- (b)  *$c$  is the largest integer with this property.*

Property (a) states that starting with any node in set  $C_k$ , the next transition must be to a node in  $C_{k+1}$ , then to a node in  $C_{k+2}$ , and so on until reaching  $C_0$ . An Example is illustrated in Fig. 1. On the other hand, if  $G$  is aperiodic ( $c = 1$ ), then there is a single set  $C_0$  containing all nodes.

**Theorem 1.** *If  $G_1$  is a strongly connected directed graph with period  $c > 2$ , then there exists a sequence of graphs  $G_2, G_3, \dots$  such that a dynamic graph  $\mathcal{G} = G_1, G_2, G_3, \dots$  is unexplorable by a simple random walk.*

*Proof.* We prove the theorem by construction. By Proposition 1, a graph  $G_t$  can be partitioned in  $c$  sets,  $C_0, C_1, \dots, C_{c-1}$ . The adversary uses the deterministic strategy of changing the direction of all the edges in each round. Let us assume that a random walk starts on some vertex  $v_k \in C_i$ , in the next round the random walk will necessarily be in the set  $C_{i+1 \bmod c}$ . As in the next round the direction of all the edges will change, the random walk will move back to some vertex in set  $C_i$  and the process repeats. In this way, only the nodes in  $C_i$  and  $C_{i+1 \bmod c}$  are reachable by the random walk that starts in  $C_i$ . □

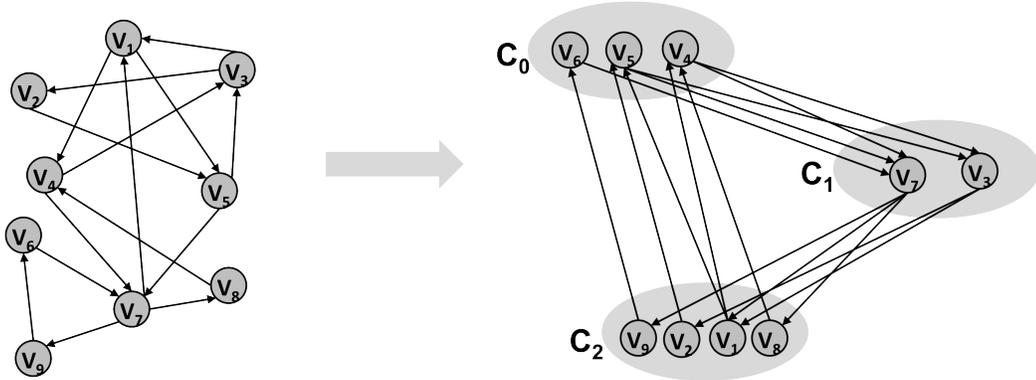


Fig. 1: Periodic Graph

### 3.2 Explorability with Lazy Random Walk

We now address the problem of whether by changing slightly the random walk strategy we can guarantee the explorability without constraining the strength of the adversary. We first observe that a lazy random walk on a graph is equivalent to a simple random walk on the same graph augmented with a self-loop at each vertex. Note that by adding self-loops in each vertex we guarantee the aperiodicity of the graph. Therefore, we avoid the negative result above. This raises the question if the strategy of *lazy random walk* is sufficient to guarantee the explorability of a dynamic graph. In the following, we answer this question positively.

**Theorem 2.** *In a directed strongly connected dynamic graph  $\mathcal{G} = G_1, G_2, \dots$ , there exists a positive probability of a lazy random walk, starting at any vertex, to reach any other vertex, within a linear number of steps.*

*Proof.* Let  $A_{G_t}$  be the transition probability matrix of a lazy random walk on  $G_t$  and  $P_t = (p_1, p_2, \dots, p_n)$  be a probability distribution on vertices in round  $t$ .

$$P_{t+1} = P_t A_{G_t} = (p_1, p_2, \dots, p_n) \begin{pmatrix} a_{1,1} & a_{1,2} & \dots & a_{1,n} \\ a_{2,1} & a_{2,2} & \dots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n,1} & a_{n,2} & \dots & a_{n,n} \end{pmatrix}$$

We denote by  $S_t$  a set of vertices that can be reached in round  $t$  that corresponds to positive entries in  $P_t$ . Without loss of generality, let us assume the random walk starts at  $v_1$  so that  $P_0 = (1, 0, \dots, 0)$  and  $S_0 = \{v_1\}$ . Note that  $a_{j,j} \geq 0$ , for all  $1 \leq j \leq n$ ; so that all positive entries in  $P_t$  will remain positive in  $P_{t+1}$ . That means that  $S_t \subseteq S_{t+1}$  for all  $t \geq 0$ . On the other hand, as in each round  $G_t$  is strongly connected, there exists an edge from some vertex  $v_i \in S_t$  to  $v_j \notin S_t$  that corresponds to a positive entry  $a_{i,j}$  in  $A_{G_t}$ . This means that  $p_j$  in  $P_{t+1}$  will become positive. Hence, in each round  $t$  at least one new vertex is added to  $S_t$ , while  $S_t \subset V$ . As a result, after  $n$  steps all entries in  $P_t$  are positive and  $S_t = V$ . □

### 3.3 Cover Times

We have shown that a directed dynamic graph is explorable with a lazy random walk. However, as we mentioned earlier, the general bound for the cover time of a random walk on a static directed graph is exponential. In Theorem 2 we showed that with lazy random walk there exists a positive probability to reach any vertex. However, the adversary still can keep it exponentially low.

We are now interested in identifying dynamic graphs with polynomial cover times. In the following, we give the upper bound for the class of directed balanced graphs (that also include undirected graphs). We say the node

$v_i$  of graph  $G$  is balanced if and only if its in-degree and out-degree are equal. A graph  $G$  is called balanced if and only if all of its nodes are balanced. The network depicted in Fig. 1 is an example of a balanced graph.

In our result, we use the following random walk strategy: Let  $d_{max}$  be a maximum degree of graph  $G$ . We define a *maximum-degree lazy random walk* as a lazy random walk where the weight of the self-loop is  $k = 1 - \frac{d}{d_{max}+1}$  and the transition probability of  $\frac{1}{d_{max}+1}$  is assigned to each outgoing edge. If  $d_{max}$  is not known, we can take  $d_{max} = n - 1$ .

To simplify the notation, we now drop the subscript  $t$ , denoting the round. The important observation is that due to maximum-degree lazy strategy, the transition probability matrices  $A_G$  in each round are doubly stochastic (every row sums to one and every column sums to one). Therefore, each  $A_G$  will have a left eigenvector  $\frac{\mathbb{1}}{n} = (\frac{1}{n}, \frac{1}{n}, \dots)$  corresponding to an eigenvalue  $\lambda = 1$ .

In the proof of our result, we will use the following lemmas. Lemma 1 bounds the absolute values of eigenvalues of  $A_G$ . Lemma 2 bounds the convergence rate of the probability distribution to the uniform distribution. Lemma 3 establishes the relation between the probability distribution on vertices of  $G$  and the cover time of a random walk on  $G$ .

**Lemma 1.** *Let  $A_G$  be the transition probability matrix of the maximum-degree lazy random walk on a balanced directed graph with  $n$  vertices. If  $|\lambda_1| \geq \dots \geq |\lambda_n|$  are the left eigenvalues of  $A_G$ , then*

$$\lambda_1 = 1 \text{ and } |\lambda_i| \leq \left(1 - \frac{1}{n^2 d_{max}}\right), \text{ for } i \geq 2$$

*Proof.* Let  $\lambda_*$  denote the spectral gap (i.e. the difference between the largest and the second largest eigenvalues in absolute value) of  $A_G$ . As shown in [Mon09],

$$\lambda_* \geq \frac{1}{n^2 d_{max}}$$

Let  $|\lambda_1| > \dots > |\lambda_n|$  be left eigenvalues of  $A_G$ . As  $A_G$  is a doubly stochastic matrix, and  $\frac{\mathbb{1}}{n} = (\frac{1}{n}, \dots, \frac{1}{n})$  is a left eigenvector of  $A_G$  corresponding to the eigenvalue  $\lambda_1 = 1$ . Moreover, as the random walk is lazy,  $A_G$  is aperiodic. Therefore, we have

$$\lambda_1 > |\lambda_i|, \text{ for } i \geq 2$$

Thus,

$$\lambda_* \geq 1 - |\lambda_i| \geq \frac{1}{n^2 d_{max}}, \text{ for } i \geq 2$$

□

**Lemma 2.** *Let  $G$  be a strongly connected directed balanced graph on  $V$  and  $P = (p_1, \dots, p_n)$  be a probability distribution on its vertices. Let  $A_G$  be a transition probability matrix of a maximum-degree lazy random walk on  $G$ . Then:*

$$\left\| PA_G - \frac{\mathbb{1}}{n} \right\|_2^2 \leq \left(1 - \frac{1}{n^2 d_{max}}\right) \left\| P - \frac{\mathbb{1}}{n} \right\|_2^2 \tag{1}$$

*Proof.* Let  $A = \{\alpha_1, \dots, \alpha_n\}$  be an orthonormal set of left eigenvectors of  $A_G$  with corresponding eigenvalues  $\lambda_1, \dots, \lambda_n \in \mathbb{C}$  ordered by their absolute values so that  $|\lambda_1| \geq \dots \geq |\lambda_n|$ . As  $A_G$  is a doubly stochastic matrix,

$$\begin{aligned} \left\| PA_G - \frac{\mathbb{1}}{n} \right\|_2^2 &= \left\| PA_G - \frac{\mathbb{1}}{n} A_G \right\|_2^2 \\ &= \left\| \left(P - \frac{\mathbb{1}}{n}\right) A_G \right\|_2^2 \end{aligned}$$

(2)

Since  $A = \{\alpha_1, \dots, \alpha_n\}$  is an orthonormal system in  $\mathbb{C}^n$  and  $(P - \frac{\mathbb{I}}{n})$  is orthogonal to  $\alpha_1 = \frac{\mathbb{I}}{\sqrt{n}}$ , there exists some  $B = \{\beta_1, \dots, \beta_n\} \in \mathbb{C}^n$  such that

$$P - \frac{\mathbb{I}}{n} = \sum_{i=2}^n \beta_i \alpha_i \tag{3}$$

By standard calculation we have,

$$\begin{aligned} \left\| P - \frac{\mathbb{I}}{n} \right\|_2^2 &= \left\langle P - \frac{\mathbb{I}}{n}, P - \frac{\mathbb{I}}{n} \right\rangle \\ &= \left\langle \sum_{i=2}^n \beta_i \alpha_i, \sum_{i=2}^n \beta_i \alpha_i \right\rangle \\ &= \sum_{i=2}^n |\beta_i|^2 \end{aligned} \tag{4}$$

On the other hand,

$$\begin{aligned} \left\| (P - \frac{\mathbb{I}}{n}) A_G \right\|_2^2 &= \left\| \sum_{i=2}^n \beta_i \alpha_i A_G \right\|_2^2 \\ &= \left\| \sum_{i=2}^n \lambda_i \beta_i \alpha_i \right\|_2^2 \\ &\leq \sum_{i=2}^n |\lambda_i|^2 |\beta_i|^2 \end{aligned} \tag{5}$$

The last inequality follows from Cauchy-Schwartz bound.

By Lemma 1,

$$|\lambda_i|^2 \leq |\lambda_i| \leq \left(1 - \frac{1}{n^2 d_{max}}\right), \text{ for } i \geq 2$$

Thus,

$$\left\| (P - \frac{\mathbb{I}}{n}) A_G \right\|_2^2 \leq \left(1 - \frac{1}{n^2 d_{max}}\right) \sum_{i=2}^n |\beta_i|^2 \tag{6}$$

From (4) and (6) we have,

$$\left\| (P - \frac{\mathbb{I}}{n}) A_G \right\|_2^2 \leq \left(1 - \frac{1}{n^2 d_{max}}\right) \left\| P - \frac{\mathbb{I}}{n} \right\|_2^2 \tag{7}$$

□

**Lemma 3.** (From [AKL08]) Let  $Y_0, Y_1, Y_2, \dots$  be a sequence of random variables with range  $V = \{v_1, \dots, v_n\}$  satisfying for all  $v_i, v_j \in V$  and  $t > 0$ ,  $Pr[Y_t = v_j | Y_{t-1} = v_i] \geq \frac{1}{2n}$ . If  $t_{min} = \min\{t : \{Y_0, Y_1, \dots, Y_t\} = V\}$  then the expectation  $E[t_{min}] \leq 3n \ln^2 n + O(\sqrt{n} \ln^2 n)$ .

Now we are ready to state our main result.

**Theorem 3.** Let  $\mathcal{G} = G_1, G_2, \dots$  be a strongly connected balanced directed dynamic graph with maximum degree  $d_{max}$ . The cover time of a maximum-degree lazy random walk on  $\mathcal{G}$  is  $O(d_{max}n^3 \ln^2 n)$ .

*Proof.* To prove this theorem we use the same technique as in [AKL08]. Let  $X_0, X_1, \dots$  be a random walk on  $\mathcal{G}$ . For an integer  $t \geq 0$ , define  $Y_t = X_{td_{max}n^2 \lceil \ln n \rceil}$ . Fix some  $v_i, v_j \in V$ . Let  $P$  be a probability distribution of  $Y_t | \{Y_{t-1} = v_i\}$ . By Lemma 2,

$$\begin{aligned} \left\| P - \frac{\mathbb{I}}{n} \right\|_2^2 &\leq \left( 1 - \frac{1}{n^2 d_{max}} \right)^{4n^2 d_{max} \ln n} \\ &< \frac{1}{n^4} \end{aligned} \tag{8}$$

Hence, the coordinates of  $P - \frac{\mathbb{I}}{n}$  are in absolute value smaller than  $\frac{1}{n^2}$ . Thus,

$$Pr[Y_t = v_j | Y_{t-1} = v_i] \geq \frac{1}{n} - \frac{1}{n^2} \geq \frac{1}{2n}, \text{ for } n \geq 2$$

Applying Lemma 3,

$$E[\min\{t : \{Y_0, Y_1, \dots, Y_t\} = V\}] = O(d_{max}n^3 \ln^2 n)$$

□

## 4 Conclusions

In this paper we addressed the problem of random walk on directed dynamic graphs. We proved that directed dynamic graphs are explorable by a lazy random walk. We also established an  $O(d_{max}n^3 \log^2 n)$  upper bound for the cover time of the maximum-degree lazy random walk on balanced directed dynamic graphs. This result also tightens the previous bound for undirected dynamic graphs, that are a special case of balanced directed graphs. We are currently working on the question of bounding the negative impact of an oblivious adversary for the entire range of directed dynamic graphs. We would also like to find out if there exists a strategy with polynomial cover time on all directed dynamic graphs.

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